A REMARK ON THE NON-COMPACTNESS OF $W^{2,d}$ IMMERSIONS OF $d$-DIMENSIONAL HYPERSURFACES

SIRAN LI

Abstract. We consider the continuous $W^{2,d}$ immersions of $d$-dimensional hypersurfaces in $\mathbb{R}^{d+1}$ with second fundamental forms uniformly bounded in $L^p$. Two results are obtained: first, a family of such immersions is constructed, whose limit fails to be an immersion of a manifold. This addresses the endpoint cases in J. Langer [6] and P. Breuning [1]. Second, under the additional assumption that the Gauss map is slowly oscillating, we prove that any family of such immersions subsequentially converges to a set locally parametrised by Hölder functions.

1. Introduction

In [6] J. Langer proved the following result: denote by $\mathcal{F}(A, E, p)$ the moduli space of immersed surfaces $\psi: \mathcal{M} \rightarrow \mathbb{R}^3$ with $\text{Area}(\psi) \leq A$, $\|II\|_{L^p(\mathcal{M})} \leq E$ and $\int_\mathcal{M} \psi \, dV = 0$. Here $A, E$ are given finite numbers, $dV$ is the volume/area measure induced by $\psi$, and $p > 2$. Then, any sequence $\{\psi_j\} \subset \mathcal{F}(A, E, p)$ contains a subsequence converging in $C^1$ to an immersed surface, modulo diffeomorphisms of $\mathcal{M}$ (written as $\text{Diff}(\mathcal{M})$). It was motivated by the study of J. Cheeger’s finiteness theorems ([2], also see K. Corlette [5]) and the Willmore energy of surfaces (see e.g., Rivière [7]). In a recent paper [1], P. Breuning generalised the above result to arbitrary dimensions and co-dimensions. More precisely, denote by $\mathcal{F}(V, E, d, n)$ the space of immersions $\psi: \mathcal{M} \rightarrow \mathbb{R}^n$ where $\mathcal{M}$ is a $d$-dimensional closed manifold, $\text{Vol}(\mathcal{M}) \leq A$, $\|II\|_{L^p(\mathcal{M})} \leq E$ and the image $\psi(\mathcal{M})$ contains a fixed point. Let $A, E, dV$ be as before, let $n > d$ be an arbitrary integer, and let $p > d$. Any sequence $\{\psi_j\} \subset \mathcal{F}(V, E, d, n)$ contains a subsequence converging in $C^1$ to an immersed submanifold, modulo $\text{Diff}(\mathcal{M})$.

The above two compactness theorems on the moduli space of immersions have a crucial assumption: $p > \dim(\mathcal{M}) = d$. Indeed, the proofs in [6, 1] utilise the Sobolev–Morrey embedding $W^{1,p}(\mathbb{R}^n) \hookrightarrow C^{0,\alpha}(\mathbb{R}^n)$, where $p > n$ and $\alpha = \alpha(p, n) \in [0, 1[$. It is natural to ask about the endpoint case $p = d$, for which the Sobolev–Morrey embedding fails. In the case $p = d = 2$, J. Langer (p.227, [6]) constructed a counterexample using conformal geometry — the Möbius inversions of the Clifford torus $T_{\text{cl}}$ with respect to a sequence of points $x_j \notin T_{\text{cl}}$ approaching an outermost point (with distance measured from the centre of the embedded image of $T_{\text{cl}}$) on $T_{\text{cl}}$ cannot tend to any immersed manifold. Clearly, such counterexamples exist only in $\mathbb{R}^2 \cong \mathbb{C}$.

Our first goal of this paper is to construct a counterexample for the $p = d$ case in arbitrary dimensions. The idea is to construct a family of hypersurfaces that "spiral wildly", resembling in some sense the motion of vortex sheets in fluid dynamics. This is achieved by letting the Gauss map $n$ (i.e., the outer unit normal vectorfield) increase rapidly from 0 to a large number $N$ as we approach some fixed point $O$, and then decrease rapidly from $N$ to 0 as we leave $O$. 

Date: July 1, 2018. 

2010 Mathematics Subject Classification. 58D10. 

Key words and phrases. Immersions; Hypersurface; Chord-Arc Surface; Second Fundamental Form; Gauss Map; Compactness; Bounded Mean Oscillations (BMO); Finiteness Theorems; Riemannian Geometry.

1
To illustrate the geometric picture, we first discuss the toy model in $d = 1$, and then construct a counterexample for general $d$. Instead of using conformal geometric methods, we exploit the scaling invariance of $||\Pi||_{L^d(M)}$, which holds in arbitrary dimensions. This is the content of §2.

Our second goal is to establish an affirmative compactness result for the $p = d$ case, with the help of an additional hypothesis: the $BMO$-norm of the Gauss map $n$,

$$\|n\|_{BMO(M)} := \sup_{x \in M, R > 0} \int_{B(x, R)} |n(y) - n_{x, R}| \, dV(y),$$

is small. Throughout $B(x, R)$ denotes the geodesic ball of radius $R$ centred at $x$ in $M$, $\mathcal{d}$ the averaged integral, and $n_{x, R} := \mathcal{d}_{B(x, R)} n \, dV$. This is inspired by the works [8, 9, 10] due to S. Semmes on the chord-arc surfaces with small constant. In §3 we shall use several results in [8, 9, 10] to prove a “partial regularity” result for the weak limit: given any family of immersed hypersurfaces $\mathbb{R}^d$ (equipped with pullback metrics) in the $(d + 1)$-dimensional Euclidean space with uniformly $L^d$-bounded second fundamental forms and small $||n||_{BMO(M)}$, one may extract a subsequence whose limit can be locally parametrised by Hölder functions.

Finally, we discuss two further questions in §4.

2. A counter-example to the endpoint case $p = d$

Let us first study the toy model $d = 1$. We prove the following simple result:

**Lemma 2.1.** There exist a family of smooth curves $\{M^e\}$ each homeomorphic to $\mathbb{R}^1$, and a family of immersions $\psi^e : M^e \rightarrow \mathbb{R}^2$ as planar curves, such that the second fundamental forms $\{\Pi^e\}$ associated to $\{\psi^e\}$ are uniformly bounded in $L^1$, but $\psi^e \circ \sigma^e$ does not converge in $C^1$-topology to any immersion of $\mathbb{R}$ for arbitrary $\{\sigma^e\} \subset \text{Diff}(\mathbb{R})$.

**Proof.** Let $J \in C^\infty_c(\mathbb{R})$ be a standard symmetric mollifier; e.g.,

$$J(s) := \Lambda \exp \left\{ \frac{1}{s^2 - 1} \right\} 1_{|s| < 1},$$

where the universal constant $\Lambda > 0$ is chosen such that $\int_{\mathbb{R}} J(s) \, ds = 1$. As usual $J^e(s) := \epsilon^{-1} J(s/\epsilon)$ for $\epsilon > 0$; then $||J_\epsilon||_{L^1(\mathbb{R})} = 1$ for every $\epsilon > 0$. In addition, define the kernel

$$K_\epsilon(x) := J_\epsilon(x + \epsilon) - J_\epsilon(x - \epsilon).$$

It satisfies $||K_\epsilon||_{L^1(\mathbb{R})} = 2$, $K_\epsilon \in C^\infty_c(\mathbb{R})$ and $\text{spt}(K_\epsilon) = [-2\epsilon, 2\epsilon]$; in particular, it is smooth at 0.

Now, define an angle function

$$\theta^e(x) := 10^m \cdot 2\pi \int_{-\infty}^{x} K_\epsilon(s) \, ds,$$

where $m \in \mathbb{Z}_+$ is to be determined. Then, set the Gauss map $n^e \in C^\infty(\mathbb{R}; S^1)$:

$$n^e(x) := \begin{bmatrix} \cos \theta^e(x) \\ \sin \theta^e(x) \end{bmatrix} \quad \text{for each } x \in \mathbb{R}.$$

The second fundamental form $\Pi^e$ equals to the negative of the gradient of the Gauss map:

$$||\Pi^e(x)|| = \sqrt{\left( - \sin \theta^e(x) \right) (\theta^e)'(x) \left( \cos \theta^e(x) \right) (\theta^e)'(x) } \left( \cos \theta^e(x) \right)^2 = ||(\theta^e)'(x)|| = (2\pi \cdot 10^m) K_\epsilon(x).$$

Thus, the $L^1$ norm of $\Pi^e$ is uniformly bounded by $4\pi \cdot 10^m$. 


Let $\psi^\epsilon$ be a smooth immersion that realises the Gauss map $n^\epsilon$ whose image is the unit circle $S^1$ in $\mathbb{R}^2$. For each $\eta > 0$, we may easily modify $\psi^\epsilon$ to $\tilde{\psi}^\epsilon$ such that $|\tilde{\psi}^\epsilon(x)|$ is decreasing on $]-\infty, 0]$ and increasing on $[0, \infty]$, the image of $\tilde{\psi}^\epsilon$ in $\mathbb{R}^2$ is homeomorphic to $\mathbb{R}^1$, and that

$$\|\psi^\epsilon - \tilde{\psi}^\epsilon\|_{C^{1,00}(\mathbb{R})} < \eta. \quad (2.6)$$

Indeed, notice that the image of $\psi^\epsilon| -\infty, 0]$ covers $S^1$ for $10^m$ times in the positive orientation, and the image of $\psi^\epsilon|0, \infty|$ covers $S^1$ for $10^m$ times in the negative orientation. We then choose the perturbed map $\tilde{\psi}^\epsilon$ such that

- As $x$ goes from $-\infty$ to $0$, $\tilde{\psi}^\epsilon$ wraps around the origin in a helical trajectory for $10^m$ times. Moreover, in each round $|\tilde{\psi}^\epsilon|$ decreases monotonically by $\sim 10^{-m}$;
- As $x$ increases from $0$ to $\infty$, $\tilde{\psi}^\epsilon$ “unwraps” around the origin along a helix for $10^m$ times, in each round $|\tilde{\psi}^\epsilon|$ increases monotonically by $\sim 10^{-m}$;
- For $x \in [-\infty, -2\epsilon[ \cup [2\epsilon, +\infty]$, the image of $\tilde{\psi}^\epsilon$ consists of straight line segments (“long flat tails”); hence $n^\epsilon$ stays constant on each component of $]-\infty, -2\epsilon[ \cup [2\epsilon, +\infty]$;
- Finally, the image $\tilde{\psi}^\epsilon(\mathbb{R})$ is $C^\infty$ and homeomorphic to $\mathbb{R}^1$.

In view of the above properties, one can take $m = m(\eta) \in \mathbb{Z}_+$ sufficiently large to verify (2.6).

Let us pick $\eta = \frac{1}{100}$, so $m$ is a universal constant fixed once and for all. Without loss of generality, from now on we may assume $\psi^\epsilon = \tilde{\psi}^\epsilon$. The point is to ensure that the image of $\psi^\epsilon$ in $\mathbb{R}^2$ is free of loops and “concentrates” near the origin $0 \in \mathbb{R}^2$, with Gauss map and second fundamental form arbitrarily close to those constructed in Eqs. (2.4)(2.5).

To conclude the proof, let us define $M^\epsilon$ as the homeomorphic $\mathbb{R}^1$ equipped with the pullback metric $(\psi^\epsilon)^\# \delta_{ij}$, where $\delta_{ij}$ is the Euclidean metric on the ambient space $\mathbb{R}^2$. It remains to show that the $C^1$-limit (modulo $\text{Diff}(\mathbb{R}^1)$) of $\psi^\epsilon$ as $\epsilon \to 0^+$ cannot be an immersion. Indeed, note that the topological degree satisfies

$$\deg(\psi^\epsilon| -\infty, 0]) = 10^m, \quad \deg(\psi^\epsilon|0, \infty[) = -10^m. \quad (2.7)$$

These identities are independent of $\epsilon$. Hence, if $\tilde{\psi}$ were a limiting immersion, (2.7) would have been preserved. However, $K_\epsilon \to \delta_0 - \delta_0 = 0$ as measures, so (2.3)(2.4)(2.5) imply that any pointwise subsequential limit of $\psi^\epsilon$ have zero topological degree. This contradiction completes the proof.

Three remarks are in order:

1. From (2.5) one may infer that

$$\|\Pi^\epsilon\|_{L^\infty(M^\epsilon)} = \frac{2\pi \cdot 10^m \cdot A}{\epsilon \epsilon} + \eta \to \infty \quad \text{as } \epsilon \to 0^+. \quad (2.8)$$

2. The construction in Lemma 2.1 can be localised near $0$. We can restrict $M^\epsilon$ to curves of finite $\mathcal{H}^1$ measure by removing the long tails. This recovers the volume bounds in [6, 1] (§1).

3. We can construct $\phi^\epsilon$ whose limit blows up at a countable discrete set $\{x_n\}$ by taking

$$\tilde{\theta}^\epsilon(x) := \sum_{n=1}^{\infty} 2^{-n} B(x_n, R_n)(x) \theta^\epsilon(x)$$

in place of $\theta^\epsilon(x)$, where $\{B(x_n, R_n)\}$ are disjoint for all $n$. Geometrically, the immersed images corresponding to $\tilde{\theta}^\epsilon$ are smooth curves that spiral towards the centres $x_n$ when $x < x_n$, and
then spiral away from \( x_n \) when \( x > x_n \). Near \( x_n \) the rate of motion blows up in \( L^\infty \) as \( \epsilon \to 0^+ \); nevertheless, its \( L^1 \) norm is constant.

Now let us generalise the above construction to \( d \)-dimensions:

**Theorem 2.2.** Let \( d \geq 1 \) be an integer. There exist a family of smooth manifolds \( \{ M^\epsilon \} \) each homeomorphic to \( \mathbb{R}^d \), and a family of immersions \( \psi^\epsilon : M^\epsilon \to \mathbb{R}^{d+1} \) as smooth hypersurfaces, such that the second fundamental forms \( \{ II^\epsilon \} \) associated to \( \{ \psi^\epsilon \} \) are uniformly bounded in \( L^d \), but \( \{ \psi^\epsilon \circ \sigma^\epsilon \} \) does not converge in \( C^1 \)-topology to any immersion of \( \mathbb{R}^d \) for arbitrary \( \{ \sigma^\epsilon \} \subset \text{Diff}(\mathbb{R}^d) \).

**Proof.** Again the crucial point is to construct the Gauss map \( n^\epsilon \in C^\infty(\mathbb{R}^d; \mathbb{S}^d) \). We make use of the spherical coordinates on \( \mathbb{S}^d \). For \( x = (x_1, x_2, \ldots, x_d) \in \mathbb{R}^d \), one needs to specify the angle functions \( \theta_i^\epsilon : \mathbb{R}^d \to \mathbb{S}^d \) for each \( i \in \{1, 2, \ldots, d\} \) in the following:

\[
\mathbf{n}^\epsilon(x) = \begin{bmatrix}
\cos \theta_1^\epsilon(x) \\
\sin \theta_1^\epsilon(x) \cos \theta_2^\epsilon(x) \\
\sin \theta_1^\epsilon(x) \sin \theta_2^\epsilon(x) \cos \theta_3^\epsilon(x) \\
\vdots \\
\sin \theta_1^\epsilon(x) \cdots \sin \theta_{d-1}^\epsilon(x) \cos \theta_d^\epsilon(x) \\
\sin \theta_1^\epsilon(x) \cdots \sin \theta_{d-1}^\epsilon(x) \sin \theta_d^\epsilon(x)
\end{bmatrix}.
\] (2.8)

Throughout we view \( \mathbb{S}^d = \{ z \in \mathbb{R}^{d+1} : |z| = 1 \} \) as the round sphere.

Indeed, let us choose

\[
\theta_i^\epsilon(x) \equiv \Theta^\epsilon(x_i) := 10^m \cdot 2\pi \int_{-\infty}^{x_i} K_i(s) \, ds,
\] (2.9)

where the kernel \( K_i \) is defined as in (2.2), and \( m \in \mathbb{Z}_+ \) is a large universal constant fixed later. Each \( \theta_i^\epsilon \) is a function of \( x_i \) only. One can easily compute all the entries in \( -II^\epsilon = \nabla \mathbf{n}^\epsilon \), which is a lower-triangular \( d \times (d + 1) \) matrix due to the embedding \( \mathbb{S}^d \hookrightarrow \mathbb{R}^{d+1} \). The rows \( \{ r_i \}_{i=1,2,\ldots,d} \) of \( \{ \nabla \mathbf{n}^\epsilon \} \) are:

\[
r_1 = \left( - (\Theta^\epsilon)'(x_1) \sin \Theta^\epsilon(x_1), 0, \ldots, 0 \right),
\]

\[
r_2 = \left( (\Theta^\epsilon)'(x_1) \cos \Theta^\epsilon(x_1) \cos \Theta^\epsilon(x_2), 0, \ldots, 0 \right),
\]

\[
r_3 = \left( (\Theta^\epsilon)'(x_1) \sin \Theta^\epsilon(x_1) \sin \Theta^\epsilon(x_2) \cos \Theta^\epsilon(x_3), \Theta^\epsilon(x_2) \sin \Theta^\epsilon(x_1) \cos \Theta^\epsilon(x_2) \cos \Theta^\epsilon(x_3), \right.
\]

\[
- (\Theta^\epsilon)'(x_3) \sin \Theta^\epsilon(x_1) \sin \Theta^\epsilon(x_2) \sin \Theta^\epsilon(x_3), 0, \ldots, 0 \right)
\]

so on and so forth, with the last two being

\[
r_{d-1} = \left( (\Theta^\epsilon)'(x_1) \cos \Theta^\epsilon(x_1) \sin \Theta^\epsilon(x_2) \cdots \sin \Theta^\epsilon(x_{d-1}) \cos \Theta^\epsilon(x_d), \ast, \ldots, \ast, \right.
\]

\[
(\Theta^\epsilon)'(x_{d-1}) \sin \Theta^\epsilon(x_1) \sin \Theta^\epsilon(x_2) \cdots \cos \Theta^\epsilon(x_{d-1}) \cos \Theta^\epsilon(x_d),
\]

\[
- (\Theta^\epsilon)'(x_d) \sin \Theta^\epsilon(x_1) \sin \Theta^\epsilon(x_2) \cdots \sin \Theta^\epsilon(x_{d-1}) \sin \Theta^\epsilon(x_d) \right)
\]

and

\[
r_d = \left( (\Theta^\epsilon)'(x_1) \cos \Theta^\epsilon(x_1) \sin \Theta^\epsilon(x_2) \cdots \sin \Theta^\epsilon(x_{d-1}) \sin \Theta^\epsilon(x_d), \ast, \ldots, \ast, \right.
\]

\[
(\Theta^\epsilon)'(x_{d-1}) \sin \Theta^\epsilon(x_1) \sin \Theta^\epsilon(x_2) \cdots \cos \Theta^\epsilon(x_{d-1}) \sin \Theta^\epsilon(x_d),
\]

\[
(\Theta^\epsilon)'(x_d) \sin \Theta^\epsilon(x_1) \sin \Theta^\epsilon(x_2) \cdots \sin \Theta^\epsilon(x_{d-1}) \cos \Theta^\epsilon(x_d) \right). \]
A tedious yet straightforward computation yields the Hilbert–Schmidt norm of the above matrix:

$$|\Pi'| = |\nabla n'| = \left| (\Theta')'(x_1), \ldots, (\Theta')'(x_d) \right|. \quad (2.10)$$

Thus, in view of (2.9) and Fubini’s theorem, we have

$$\|\Pi\|_{L^d(\mathbb{R}^d)} = 10^m \cdot 2\pi \left\| K_{i_1} \otimes \cdots \otimes K_{i_d} \right\|_{L^d(\mathbb{R}^d)} = 10^m \cdot 2\pi \| K_{i_1} \|_{L^1(\mathbb{R}^d)} = 10^m \cdot 4\pi. \quad (2.11)$$

Now we shall choose a smooth immersion that (approximately) realises $n'$ precisely as in the $d = 1$ case (Lemma 2.1). For the sake of completeness let us sketch the arguments. First, take $\psi^\epsilon$ whose Gauss map is $n'$ and which takes value in $S^d$. Then we may modify it — without relabelling and up to an arbitrarily small error, say $\frac{1}{100}$ in the $C^{100}$-topology — so that the image of $\psi^\epsilon$ in $\mathbb{R}^{d+1}$ is a smooth, homeomorphic copy of $\mathbb{R}^d$ for each $\epsilon > 0$, having flat ends outside $B(0, 2)$, and having $d$ independent angle functions in the spherical coordinates (i.e., in place of $\theta_i^\epsilon$’s in (2.8)) wrapping around $0 \in \mathbb{R}^{d+1}$ for $10^m$ times in the positive orientation and unwrapping for $10^m$ times in the negative orientation. The second fundamental form of the modified map $\psi^\epsilon$ satisfies the bound in (2.11), up to an error of $\pm \frac{1}{100}$. Then, define $\mathcal{M}^\epsilon := ([\mathbb{R}^d, (\psi^\epsilon)^\# \delta_{ij})$, where $\delta_{ij}$ is the Euclidean metric on $\mathbb{R}^{d+1}$. By a topological degree argument as in (2.7), the limit of $\psi^\epsilon$ cannot be an immersion up to the action of $\text{Diff}(\mathbb{R}^n)$. This completes the proof. \(\square\)

Similar to the remarks below the proof of Lemma 2.1, this counterexample can be localised, and an iteration yields a family of immersions of $\mathbb{R}^d$ that blows up at an infinite discrete set.

3. Local Hölder Regularity

In this section we deduce a compactness theorem utilising the works [8, 9, 10] of S. Semmes on the harmonic analysis on chord-arc surfaces with small constants. Consider the moduli space

$$\mathcal{F}(\delta, d) := \left\{ f \in W^{2,d} \cap C^{\infty}(\mathcal{M}^\epsilon; \mathbb{R}^{d+1}) : f \text{ is an immersion}, \mathcal{M} \text{ is an } d\text{-dimensional hypersurface}, \mathcal{M} \cup \{\infty\} \text{ is smooth in } S^{d+1}, \|n\|_{\text{BMO}(\mathcal{M})} \leq \delta, f(\mathcal{M}) \text{ contains a fixed point} \right\}. \quad (3.1)$$

We show the following: if the Gauss maps of a family of smooth homeomorphic $\mathbb{R}^d$ have uniformly small oscillations at all scales, then “a little” regularity persists in the limit. This assumption is natural: if a family of $W^{2,d}$ immersions of $d$-manifolds has uniformly $L^d$-bounded second fundamental forms, then their Gauss maps have bounded BMO-norms (provided that Poincaré and Sobolev inequalities hold).

For this purpose we need a definition. A set $\Omega \subset \mathbb{R}^d$ is called a Hölder graph system if it can be locally represented by graphs of $C^{0,\gamma}$ functions for some $\gamma \in [0, 1]$. We do not require further geometric information for a Hölder graph system, e.g., whether or not it represents a topological manifold or orbifold. The notion of “graph system” plays an essential role in [6, 1] by J. Langer and P. Breuning.

**Theorem 3.1.** There exists a small constant $\delta_0 > 0$ depending only on the dimension $d$, such that for any $\delta \in [0, \delta_0]$ and any family of immersions $\{\psi^\epsilon\} \subset \mathcal{F}(d, \delta)$, we can find $\{\sigma^\epsilon\} \subset \text{Diff}(\mathbb{R}^d)$ such that the limit of $\psi^\epsilon \circ \sigma^\epsilon$ converges to a Hölder graph system, after passing to subsequences.
It is proved in [9, 10] that for sufficiently small $\delta_0$, $\mathcal{M}$ is homeomorphic to $\mathbb{R}^d$ and behaves nicely on small scales — for each $x \in \mathcal{M}$ and $R > 0$, $B(x, R) \cap \mathcal{M}$ stays close to the hyperplane through $x$ normal to the averaged Gauss map $\mathbf{n}_{x,R}$. Indeed, $\mathcal{M}$ with small $\|\mathbf{n}\|_{\text{BMO}(\mathcal{M})}$ is equivalent to the definition of a chord-arc surface with small constant, defined in [8] as a generalisation of the chord-arc domain for $d = 1$. Although it remains an open question if such $\mathcal{M}$ always admits bi-Lipschitz parametrisations by $\mathbb{R}^d$ (cf. T. Toro [11] for a related problem), it is nevertheless known that $\mathcal{M}$ has a “bi-Hölder” parametrisation; see Theorem 4.1, [9]. This enables us to prove Theorem 3.1.

Proof. Let us first summarise several estimates from [8, 9, 10]. Fix any $t > 0$, e.g. $t = 10^{-5}$. By §3, [9] one can find a new chord-arc surface $\mathcal{M}_t$ with the chord-arc constant $\mu$, such that

$$0 \leq \delta \leq \delta_0 \leq C(d)\delta_0 < \mu.$$  

We shall choose $\mu$ later, which is equivalent to the least upper bound for the $\text{BMO}$-norm of the Gauss map; see p.200 [8]. In view of Eq. (3.7) and Lemma 3.8 in [9], $\mathcal{M}_t \cap B(x, [2^{-1} + 10^{-10}]t)$ is a Lipschitz graph with constant $\leq C_0 \mu$ for each $x \in \mathcal{M}$, provided that $\mu = \mu(t, \delta_0)$ is chosen large enough. Here $C_0 = C(d, \delta_0)$. Under the same condition, $\mathcal{M}_t$ can be taken sufficiently close to $\mathcal{M}$ (e.g., with distance $\leq 10^{-10}t$ by Lemma 3.8 in [9]). Then, in view of Theorem 4.1 in [9], there exists a homeomorphism $\tau: \mathcal{M} \to \mathcal{M}_t$ such that

$$\max \left\{ \|\tau\|_{C^{0,\gamma}(B(x,10000\gamma \mathcal{M}))}, \|\tau^{-1}\|_{C^{0,\gamma}(B(x,10000\gamma \mathcal{M}))} \right\} \leq C_1 \quad \text{for all } x \in \mathcal{M},$$  

(3.2)

where $C_1 = C(d, \delta_0, t)$ and the Hölder index is given by

$$\gamma \equiv 1 - C_2 d \delta_0$$  

(3.3)

for a dimensional constant $C_2$ (denoted by $k$ in [9]). In fact, putting together Eqs. (1.3)(4.6) and the choice of $p$ on p.178 in [9], Lemma 5.5 in [8] and that $0 \leq \delta \leq \delta_0$, we may explicitly select

$$C_1 = C_3^C_2 \delta_0 \left\{ \frac{(100t)^C_2 \delta_0}{1 - 2 \cdot 10^4 \delta_0} \right\}.$$  

(3.4)

Here $C_3 = C_3(d)$ is a dimensional constant. Notice that our estimates (3.4)(3.2) are uniform in $\delta$. We also have to further restrict to $\delta_0 < (C_2 d)^{-1}$ to ensure that $\gamma > 0$ in (3.3).

Now we are ready to give the proof. By considering a compact exhaustion $\{\mathcal{M}_k\} \not\supset \mathcal{M}$, one may take $\mathcal{M}$ to be a bounded domain in $\mathbb{R}^d$. (The argument for non-compact manifolds in the $p > d$ case is more involved, if one needs to check that the limiting object is a manifold; see §7 in [1].) Then we can take a $(50t)$-net $\mathcal{N}$ of $\mathcal{M}$, whose cardinality is

$$\mathcal{H}^0(\mathcal{N}) = C_4 t^{-d}$$

for some geometric constant $C_4 = C(d, \gamma) \equiv C(d, \delta_0)$. In each element of $\mathcal{N}$ the hypersurface $\mathcal{M}$ is $C^{0,\gamma}$-parametrised by $\mathcal{M}_t$, which is a Lipschitz graph on $(2^{-1} + 10^{-10})$-balls. Using the quantitative estimates in the preceding paragraph, we can refine $\mathcal{N}$ to a sub-net $\tilde{\mathcal{N}}$ with cardinality $C_5 t^{-d}$, $C_5 = C(d, \delta_0)$ again, such that in each $B \in \tilde{\mathcal{N}}$, the set $B \cap \mathcal{M}$ is parametrised by a $C^{0,\gamma}$-homeomorphism with the Hölder norm bounded by $C_6 := C_0 \mu \cdot C_1$. Let us choose $\mu = 10C(d)\delta_0$; then $C_6 = C(d, \delta_0, t)$ (where $t > 0$ is fixed from the beginning). Therefore, in view of the Arzelà–Ascoli theorem, i.e., the compactness of he embedding $C^{0,\gamma} \to C^{0,\gamma'}$ for $\gamma' \in]0, \gamma[$ and the uniform estimates derived above, we may complete the proof. \(\square\)
4. Two Further Questions

Let the moduli space \( \mathcal{F}(A, E, p) \) be as in §1. Is the space
\[
\mathcal{F}_{\text{isom}}(A, E, p) := \left\{ \psi \in \mathcal{F}(A, E, p) : \psi \text{ is an isometric immersion of a fixed manifold } M \right\}
\]
compact in its natural topology? For the end-point case \( p = 2 = d \) the answer is affirmative, in contrast to the unconstrained case for \( \mathcal{F}(A, E, p) \). The authors of [3] proved this via establishing the weak continuity of the Gauss–Codazzi equations (the PDE system for the isometric immersion), with the help of a div-curl type lemma due to Conti–Dolzmann–Müller in [4]. What about higher dimensions \( d \geq 3 \) (and co-dimensions greater than 1)? That is, for a family of isometric immersions of some fixed \( d \)-dimensional manifold with uniformly bounded second fundamental forms in \( L^d \), is the subsequential limit an isometric immersion?

Theorem 3.1 leaves open the possibility that the limiting objects of \( W^{2,d} \)-bounded immersed hypersurfaces may be very irregular (e.g., the nowhere differentiable Weierstrass function is \( C^{0,\gamma} \), or other fractals), even if the (somewhat strong) geometrical condition that the Gauss map is slowly oscillating is enforced. Can we find natural geometrical conditions on the moduli space of \( d \)-dimensional hypersurfaces with uniformly bounded second fundamental forms in \( L^d \), which is sufficient to ensure higher regularities for the subsequential limits, e.g., BV or Lipschitz? This is related to the problem of finding good parametrizations of chord-arc surfaces; see the discussions by S. Semmes [9] and T. Toro [11].

Acknowledgement. This work has been done during the author’s stay as a CRM–ISM post-doctoral fellow at the Centre de Recherches Mathématiques, Université de Montréal and the Institut des Sciences Mathématiques. I would like to thank these institutions for their hospitality. I am also indebted to Prof. Gui-Qiang Chen for his continuous support and many insightful discussions on weak compactness of (isometric) immersions.

References

Siran Li, Department of Mathematics, Rice University, MS 136 P.O. Box 1892, Houston, Texas, 77251-1892, USA; Department of Mathematics, McGill University, Burnside Hall, 805 Sherbrooke Street West, Montreal, Quebec, H3A 0B9, Canada.

E-mail address: Siran.Li@rice.edu; siran.li@mail.mcgill.ca