A REMARK ON THE NON-COMPACTNESS OF $W^{2,d}$ IMMERSIONS OF d-DIMENSIONAL HYPERSURFACES

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Abstract. We consider the continuous $W^{2,d}$ immersions of d-dimensional hypersurfaces in $\mathbb{R}^{d+1}$ with second fundamental forms uniformly bounded in L^d. Two results are obtained: first, a family of such immersions is constructed, whose limit fails to be an immersion of a manifold. This addresses the endpoint cases in J. Langer [6] and P. Breuning [1]. Second, under the additional assumption that the Gauss map is slowly oscillating, we prove that any family of such immersions subsequentially converges to a set locally parametrised by Hölder functions.

1. Introduction

In [6] J. Langer proved the following result: denote by $\mathcal{F}(A,E,p)$ the moduli space of immersed surfaces $\psi : \mathcal{M} \to \mathbb{R}^3$ with $\mathrm{Area}(\psi) \leq A$, $\|\mathrm{II}\|_{L^p(\mathcal{M})} \leq E$ and $\int_{\mathcal{M}} \psi \,\mathrm{d}V = 0$. Here A, E are given finite numbers, $\mathrm{d}V$ is the volume/area measure induced by ψ, and $p > 2$. Then, any sequence $\{\psi_j\} \subset \mathcal{F}(A,E,p)$ contains a subsequence converging in C^1 to an immersed surface, modulo diffeomorphisms of $\mathcal{M}$ (written as $\mathrm{Diff}(\mathcal{M})$). It was motivated by the study of J. Cheeger's finiteness theorems ([2], also see K. Corlette [5]) and the Willmore energy of surfaces (see *e.g.*, Rivière [7]). In a recent paper [1], P. Breuning generalised the above result to arbitrary dimensions and co-dimensions. More precisely, denote by $\mathcal{F}(V,E,d,n)$ the space of immersions $\psi : \mathcal{M} \to \mathbb{R}^n$ where $\mathcal{M}$ is a d-dimensional closed manifold, $Vol(\mathcal{M}) \leq A$, $\|\mathrm{II}\|_{L^p(\mathcal{M})} \leq E$ and the image $\psi(\mathcal{M})$ contains a fixed point. Let $A, E, \mathrm{d}V$ be as before, let $n > d$ be an arbitrary integer, and let $p > d$. Any sequence $\{\psi_j\} \subset \mathcal{F}(V,E,d,n)$ contains a subsequence converging in C^1 to an immersed submanifold, modulo $\mathrm{Diff}(\mathcal{M})$.

The above two compactness theorems on the moduli space of immersions have a crucial assumption: $p > \dim(\mathcal{M}) = d$. Indeed, the proofs in [6, 1] utilise the Sobolev–Morrey embedding $W^{1,p}(\mathbb{R}^n) \hookrightarrow C^{0,\alpha}(\mathbb{R}^n)$, where $p > n$ and $\alpha = \alpha(p,n) \in]0,1[$. It is natural to ask about the end-point case $p = d$, for which the Sobolev–Morrey embedding fails. In the case $p = d = 2$, J. Langer (p.227, [6]) constructed a counterexample using conformal geometry — the Möbius inversions of the Clifford torus T_{cl} with respect to a sequence of points $x_j \notin T_{\mathrm{cl}}$ approaching an outermost point (with distance measured from the centre of the embedded image of T_{cl}) on T_{cl} cannot tend to any immersed manifold. Clearly, such counterexamples exist only in $\mathbb{R}^2 \cong \mathbb{C}$.

Our first goal of this paper is to construct a counterexample for the $p = d$ case in *arbitrary* dimensions. The idea is to construct a family of hypersurfaces that "spiral wildly", resembling in some sense the motion of vortex sheets in fluid dynamics. This is achieved by letting the Gauss map $\mathbf{n}$ (*i.e.*, the outer unit normal vectorfield) increase rapidly from 0 to a large number N as we approach some fixed point O, and then decrease rapidly from N to 0 as we leave O.

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To illustrate the geometric picture, we first discuss the toy model in $d = 1$, and then construct a counterexample for general d. Instead of using conformal geometric methods, we exploit the scaling invariance of $\|\mathrm{II}\|_{L^d(\mathcal{M})}$, which holds in arbitrary dimensions. This is the content of §2.

Our second goal is to establish an affirmative compactness result for the $p = d$ case, with the help of an additional hypothesis: the BMO-norm of the Gauss map $\mathbf{n}$,

$$\|\mathbf{n}\|_{\mathrm{BMO}(\mathcal{M})} := \sup_{x\in\mathcal{M},\, R>0} \fint_{\mathcal{M}\cap B(x,R)} |\mathbf{n}(y) - \mathbf{n}_{x,R}| \,\mathrm{d}V(y), \tag{1.1}$$

is small. Throughout $B(x,R)$ denotes the geodesic ball of radius R centred are x in $\mathcal{M}$, $\fint$ the averaged integral, and $\mathbf{n}_{x,R} := \fint_{B(x,R)} \mathbf{n}\,\mathrm{d}V$. This is inspired by the works [8, 9, 10] due to S. Semmes on the chord-arc surfaces with small constant. In §3 we shall use several results in [8, 9, 10] to prove a "partial regularity" result for the weak limit: given any family of immersed hypersurfaces $\mathbb{R}^d$ (equipped with pullback metrics) in the $(d+1)$-dimensional Euclidean space with uniformly L^d-bounded second fundamental forms and small $\|\mathbf{n}\|_{\mathrm{BMO}(\mathcal{M})}$, one may extract a subsequence whose limit can be locally parametrised by Hölder functions.

Finally, we discuss two further questions in §4.

2. A COUNTER-EXAMPLE TO THE ENDPOINT CASE $p = d$

Let us first study the toy model $d = 1$. We prove the following simple result:

Lemma 2.1. *There exist a family of smooth curves $\{\mathcal{M}^\epsilon\}$ each homeomorphic to $\mathbb{R}^1$, and a family of immersions $\psi^\epsilon : \mathcal{M}^\epsilon \to \mathbb{R}^2$ as planar curves, such that the second fundamental forms $\{\mathrm{II}^\epsilon\}$ associated to $\{\psi^\epsilon\}$ are uniformly bounded in L^1, but $\{\psi^\epsilon \circ \sigma^\epsilon\}$ does not converge in C^1-topology to any immersion of $\mathbb{R}$ for arbitrary $\{\sigma^\epsilon\} \subset \mathrm{Diff}(\mathbb{R})$.*

Proof. Let $J \in C^\infty_c(\mathbb{R})$ be a standard symmetric mollifier; *e.g.*,

$$J(s) := \Lambda \exp\left\{\frac{1}{s^2-1}\right\} \mathbb{1}_{\{|s|<1\}}, \tag{2.1}$$

where the universal constant $\Lambda > 0$ is chosen such that $\int_{\mathbb{R}} J(s)\,\mathrm{d}s = 1$. As usual $J^\epsilon(s) := \epsilon^{-1} J(s/\epsilon)$ for $\epsilon > 0$; then $\|J_\epsilon\|_{L^1(\mathbb{R})} = 1$ for every $\epsilon > 0$. In addition, define the kernel

$$K_\epsilon(x) := J_\epsilon(x+\epsilon) - J_\epsilon(x-\epsilon). \tag{2.2}$$

It satisfies $\|K_\epsilon\|_{L^1(\mathbb{R})} = 2$, $K_\epsilon \in C^\infty_c(\mathbb{R})$ and $\mathrm{spt}(K_\epsilon) = [-2\epsilon, 2\epsilon]$; in particular, it is smooth at 0.

Now, define an angle function

$$\theta^\epsilon(x) := 10^m \cdot 2\pi \int_{-\infty}^x K_\epsilon(s)\,\mathrm{d}s, \tag{2.3}$$

where $m \in \mathbb{Z}_+$ is to be determined. Then, set the Gauss map $\mathbf{n}^\epsilon \in C^\infty(\mathbb{R}; \mathbb{S}^1)$:

$$\mathbf{n}^\epsilon(x) := \begin{bmatrix} \cos\theta^\epsilon(x) \\ \sin\theta^\epsilon(x) \end{bmatrix} \qquad \text{for each } x \in \mathbb{R}. \tag{2.4}$$

The second fundamental form II^ϵ equals to the negative of the gradient of the Gauss map:

$$\begin{aligned} |\mathrm{II}^\epsilon(x)| &= \sqrt{\Big|\big(-\sin\theta^\epsilon(x)\big)(\theta^\epsilon)'(x)\Big|^2 + \Big|\big(\cos\theta^\epsilon(x)\big)(\theta^\epsilon)'(x)\Big|^2} \\ &= |(\theta^\epsilon)'(x)| = (2\pi\cdot 10^m) K_\epsilon(x). \end{aligned} \tag{2.5}$$

Thus, the L^1 norm of $\{\mathrm{II}^\epsilon\}$ is uniformly bounded by $4\pi \cdot 10^m$.

Let ψ^ϵ be a smooth immersion that realises the Gauss map $\mathbf{n}^\epsilon$ whose image is the unit circle $\mathbb{S}^1$ in $\mathbb{R}^2$. For each $\eta > 0$, we may easily modify ψ^ϵ to $\tilde{\psi}^\epsilon$ such that $|\tilde{\psi}^\epsilon(x)|$ is decreasing on $]-\infty, 0]$ and increasing on $[0, \infty[$, the image of $\tilde{\psi}^\epsilon$ in $\mathbb{R}^2$ is homeomorphic to $\mathbb{R}^1$, and that

$$\|\psi^\epsilon - \tilde{\psi}^\epsilon\|_{C^{100}(\mathbb{R})} < \eta. \tag{2.6}$$

Indeed, notice that the image of $\psi^\epsilon\Big|]-\infty, 0]$ covers $\mathbb{S}^1$ for 10^m times in the positive orientation, and the image of $\psi^\epsilon\Big|[0, \infty[$ covers $\mathbb{S}^1$ for 10^m times in the negative orientation. We then choose the perturbed map $\tilde{\psi}^\epsilon$ such that

- As x goes from $-\infty$ to 0, $\tilde{\psi}^\epsilon$ wraps around the origin in a helical trajectory for 10^m times. Moreover, in each round $|\tilde{\psi}^\epsilon|$ decreases monotonically by $\sim 10^{-m}$;
- As x increases from 0 to ∞, $\tilde{\psi}^\epsilon$ "unwraps" around the origin along a helix for 10^m times, in each round $|\tilde{\psi}^\epsilon|$ increases monotonically by $\sim 10^{-m}$;
- For $x \in]-\infty, -2\epsilon] \sqcup [2\epsilon, +\infty[$, the image of $\tilde{\psi}^\epsilon$ consists of straight line segments ("long flat tails"); hence $\mathbf{n}^\epsilon$ stays constant on each component of $]-\infty, -2\epsilon] \sqcup [2\epsilon, +\infty[$;
- Finally, the image $\tilde{\psi}^\epsilon(\mathbb{R})$ is C^∞ and homeomorphic to $\mathbb{R}^1$.

In view of the above properties, one can take $m = m(\eta) \in \mathbb{Z}_+$ sufficiently large to verify (2.6). Let us pick $\eta = \frac{1}{100}$, so m is a universal constant fixed once and for all. Without loss of generality, from now on we may assume $\psi^\epsilon = \tilde{\psi}^\epsilon$. The point is to ensure that the image of ψ^ϵ in $\mathbb{R}^2$ is free of loops and "concentrates" near the origin $0 \in \mathbb{R}^2$, with Gauss map and second fundamental form arbitrarily close to those constructed in Eqs. (2.4)(2.5).

To conclude the proof, let us define $\mathcal{M}^\epsilon$ as the homeomorphic $\mathbb{R}^1$ equipped with the pullback metric $(\psi^\epsilon)^\# \delta_{ij}$, where δ_{ij} is the Euclidean metric on the ambient space $\mathbb{R}^2$. It remains to show that the C^1-limit (modulo $\mathrm{Diff}(\mathbb{R}^1)$) of ψ^ϵ as $\epsilon \to 0^+$ cannot be an immersion. Indeed, note that the topological degree satisfies

$$\deg\left(\psi^\epsilon\Big|]-\infty, 0]\right) = 10^m, \qquad \deg\left(\psi^\epsilon\Big|[0, \infty[\right) = -10^m. \tag{2.7}$$

These identities are independent of ϵ. Hence, if $\bar{\psi}$ were a limiting immersion, (2.7) would have been preserved. However, $K_\epsilon \overset{*}{\rightharpoonup} \delta_0 - \delta_0 = 0$ as measures, so (2.3)(2.4)(2.5) imply that any pointwise subsequential limit of ψ^ϵ have zero topological degree. This contradiction completes the proof. □

Three remarks are in order:

1. From (2.5) one may infer that

$$\|\mathrm{II}^\epsilon\|_{L^\infty(\mathcal{M}^\epsilon)} = \frac{2\pi \cdot 10^m \cdot \Lambda}{e\epsilon} + \eta \longrightarrow \infty \qquad \text{as } \epsilon \to 0^+.$$

2. The construction in Lemma 2.1 can be localised near 0. We can restrict $\mathcal{M}^\epsilon$ to curves of finite $\mathcal{H}^1$ measure by removing the long tails. This recovers the volume bounds in [6, 1] (§1).

3. We can construct ϕ^ϵ whose limit blows up at a countable discrete set $\{x_n\}$ by taking

$$\tilde{\theta}^\epsilon(x) := \sum_{n=1}^\infty 2^{-n} \mathbb{1}_{B(x_n, R_n)}(x) \theta^\epsilon(x)$$

in place of $\theta^\epsilon(x)$, where $\{B(x_n, R_n)\}$ are disjoint for all n. Geometrically, the immersed images corresponding to $\tilde{\theta}^\epsilon$ are smooth curves that spiral towards the centres x_n when $x < x_n$, and then spiral away from x_n when $x > x_n$. Near x_n the rate of motion blows up in L^∞ as $\epsilon \to 0^+$; nevertheless, its L^1 norm is constant.

Now let us generalise the above construction to d-dimensions:

Theorem 2.2. *Let $d \geq 1$ be an integer. There exist a family of smooth manifolds $\{\mathcal{M}^\epsilon\}$ each homeomorphic to $\mathbb{R}^d$, and a family of immersions $\psi^\epsilon : \mathcal{M}^\epsilon \to \mathbb{R}^{d+1}$ as smooth hypersurfaces, such that the second fundamental forms $\{\mathrm{II}^\epsilon\}$ associated to $\{\psi^\epsilon\}$ are uniformly bounded in L^d, but $\{\psi^\epsilon \circ \sigma^\epsilon\}$ does not converge in C^1-topology to any immersion of $\mathbb{R}^d$ for arbitrary $\{\sigma^\epsilon\} \subset \mathrm{Diff}(\mathbb{R}^d)$.*

Proof. Again the crucial point is to construct the Gauss map $\mathbf{n}^\epsilon \in C^\infty(\mathbb{R}^d; \mathbb{S}^d)$. We make use of the spherical coordinates on $\mathbb{S}^d$. For $x = (x_1, x_2, \ldots, x_d) \in \mathbb{R}^d$, one needs to specify the angle functions $\theta_i^\epsilon : \mathbb{R}^d \to \mathbb{S}^d$ for each $i \in \{1, 2, \ldots, d\}$ in the following:

$$\mathbf{n}^\epsilon(x) = \begin{bmatrix} \cos\theta_1^\epsilon(x) \\ \sin\theta_1^\epsilon(x)\cos\theta_2^\epsilon(x) \\ \sin\theta_1^\epsilon(x)\sin\theta_2^\epsilon(x)\cos\theta_3^\epsilon(x) \\ \vdots \\ \sin\theta_1^\epsilon(x)\cdots\sin\theta_{d-1}^\epsilon(x)\cos\theta_d^\epsilon(x) \\ \sin\theta_1^\epsilon(x)\cdots\sin\theta_{d-1}^\epsilon(x)\sin\theta_d^\epsilon(x) \end{bmatrix}. \tag{2.8}$$

Throughout we view $\mathbb{S}^d = \{z \in \mathbb{R}^{d+1} : |z| = 1\}$ as the round sphere.

Indeed, let us choose

$$\theta_i^\epsilon(x) \equiv \Theta^\epsilon(x_i) := 10^m \cdot 2\pi \int_{-\infty}^{x_i} K_\epsilon(s)\,\mathrm{d}s, \tag{2.9}$$

where the kernel K_ϵ is defined as in (2.2), and $m \in \mathbb{Z}_+$ is a large universal constant fixed later. Each θ_i^ϵ is a function of x_i only. One can easily compute all the entries in $-\mathrm{II}^\epsilon = \nabla\mathbf{n}^\epsilon$, which is a lower-triangular $d \times (d+1)$ matrix due to the embedding $\mathbb{S}^d \hookrightarrow \mathbb{R}^{d+1}$. The rows $\{\mathbf{r}_i\}_{i=1,2,\ldots,d}$ of $\{\nabla\mathbf{n}^\epsilon\}$ are:

$$\mathbf{r}_1 = \Big(-(\Theta^\epsilon)'(x_1)\sin\Theta^\epsilon(x_1), 0, \cdots, 0\Big),$$

$$\mathbf{r}_2 = \Big((\Theta^\epsilon)'(x_1)\cos\Theta^\epsilon(x_1)\cos\Theta^\epsilon(x_2), 0, \cdots, 0\Big),$$

$$\begin{aligned}\mathbf{r}_3 = \Big(&(\Theta^\epsilon)'(x_1)\sin\Theta^\epsilon(x_1)\sin\Theta^\epsilon(x_2)\cos\Theta^\epsilon(x_3), \Theta^\epsilon(x_2)\sin\Theta^\epsilon(x_1)\cos\Theta^\epsilon(x_2)\cos\Theta^\epsilon(x_3),\\ &-(\Theta^\epsilon)'(x_3)\sin\Theta^\epsilon(x_1)\sin\Theta^\epsilon(x_2)\sin\Theta^\epsilon(x_3), 0, \cdots, 0\Big)\end{aligned}$$

so on and so forth, with the last two being

$$\begin{aligned}\mathbf{r}_{d-1} = \Big(&(\Theta^\epsilon)'(x_1)\cos\Theta^\epsilon(x_1)\sin\Theta^\epsilon(x_2)\cdots\sin\Theta^\epsilon(x_{d-1})\cos\Theta^\epsilon(x_d), *, \cdots, *,\\ &(\Theta^\epsilon)'(x_{d-1})\sin\Theta^\epsilon(x_1)\sin\Theta^\epsilon(x_2)\cdots\cos\Theta^\epsilon(x_{d-1})\cos\Theta^\epsilon(x_d),\\ &-(\Theta^\epsilon)'(x_d)\sin\Theta^\epsilon(x_1)\sin\Theta^\epsilon(x_2)\cdots\sin\Theta^\epsilon(x_{d-1})\sin\Theta^\epsilon(x_d)\Big)\end{aligned}$$

and

$$\begin{aligned}\mathbf{r}_d = \Big(&(\Theta^\epsilon)'(x_1)\cos\Theta^\epsilon(x_1)\sin\Theta^\epsilon(x_2)\cdots\sin\Theta^\epsilon(x_{d-1})\sin\Theta^\epsilon(x_d), *, \cdots, *,\\ &(\Theta^\epsilon)'(x_{d-1})\sin\Theta^\epsilon(x_1)\sin\Theta^\epsilon(x_2)\cdots\cos\Theta^\epsilon(x_{d-1})\sin\Theta^\epsilon(x_d),\\ &(\Theta^\epsilon)'(x_d)\sin\Theta^\epsilon(x_1)\sin\Theta^\epsilon(x_2)\cdots\sin\Theta^\epsilon(x_{d-1})\cos\Theta^\epsilon(x_d)\Big).\end{aligned}$$

A tedious yet straightforward computation yields the Hilbert–Schmidt norm of the above matrix:

$$|\mathrm{II}^\epsilon| = |\nabla \mathbf{n}^\epsilon| = \Big|\Big((\Theta^\epsilon)'(x_1), \cdots, (\Theta^\epsilon)'(x_d)\Big)\Big|. \tag{2.10}$$

Thus, in view of (2.9) and Fubini's theorem, we have

$$\|\mathrm{II}\|_{L^d(\mathbb{R}^d)} = 10^m \cdot 2\pi \Big\| \underbrace{K_\epsilon \otimes \cdots \otimes K_\epsilon}_{d \text{ times}} \Big\|_{L^d(\mathbb{R}^d)} = 10^m \cdot 2\pi \, \|K^\epsilon\|_{L^1(\mathbb{R}^d)} = 10^m \cdot 4\pi. \tag{2.11}$$

Now we shall choose a smooth immersion that (approximately) realises $\mathbf{n}^\epsilon$ precisely as in the $d = 1$ case (Lemma 2.1). For the sake of completeness let us sketch the arguments. First, take ψ^ϵ whose Gauss map is $\mathbf{n}^\epsilon$ and which takes value in $\mathbb{S}^d$. Then we may modify it — without relabelling and up to an arbitrarily small error, say $\frac{1}{100}$ in the C^{100}-topology — so that the image of ψ^ϵ in $\mathbb{R}^{d+1}$ is a smooth, homeomorphic copy of $\mathbb{R}^d$ for each $\epsilon > 0$, having flat ends outside $B(0, 2)$, and having d independent angle functions in the spherical coordinates (*i.e.*, in place of θ_i^ϵ's in (2.8)) wrapping around $0 \in \mathbb{R}^{d+1}$ for 10^m times in the positive orientation and unwrapping for 10^m times in the negative orientation. The second fundamental form of the modified map ψ^ϵ satisfies the bound in (2.11), up to an error of $\pm\frac{1}{100}$. Then, define $\mathcal{M}^\epsilon := (\mathbb{R}^d, (\psi^\epsilon)^\# \delta_{ij})$, where δ_{ij} is the Euclidean metric on $\mathbb{R}^{d+1}$. By a topological degree argument as in (2.7), the limit of ψ^ϵ cannot be an immersion up to the action of $\mathrm{Diff}(\mathbb{R}^n)$. This completes the proof. $\square$

Similar to the remarks below the proof of Lemma 2.1, this counterexample can be localised, and an iteration yields a family of immersions of $\mathbb{R}^d$ that blows up at an infinite discrete set.

3. Local Hölder Regularity

In this section we deduce a compactness theorem utilising the works [8, 9, 10] of S. Semmes on the harmonic analysis on chord-arc surfaces with small constants. Consider the moduli space

$$\mathcal{F}(\delta, d) := \Big\{ f \in W^{2,d} \cap C^\infty(\mathcal{M}; \mathbb{R}^{d+1}) : f \text{ is an immersion, } \mathcal{M} \text{ is an } d\text{-dimensional hypersurface,}$$
$$\mathcal{M} \cup \{\infty\} \text{ is smooth in } \mathbb{S}^{d+1}, \|\mathbf{n}\|_{\mathrm{BMO}(\mathcal{M})} \leq \delta, \ f(\mathcal{M}) \text{ contains a fixed point} \Big\}. \tag{3.1}$$

We show the following: if the Gauss maps of a family of smooth homeomorphic $\mathbb{R}^d$ have uniformly small oscillations at all scales, then "a little" regularity persists in the limit. This assumption is natural: if a family of $W^{2,d}$ immersions of d-manifolds has uniformly L^d-bounded second fundamental forms, then their Gauss maps have bounded BMO-norms (provided that Poincaré and Sobolev inequalities hold).

For this purpose we need a definition. A set $\Omega \subset \mathbb{R}^d$ is called a *Hölder graph system* if it can be locally represented by graphs of $C^{0,\gamma}$ functions for some $\gamma \in]0, 1]$. We do not require further geometric information for a Hölder graph system, *e.g.*, whether or not it represents a topological manifold or orbifold. The notion of "graph system" plays an essential role in [6, 1] by J. Langer and P. Breuning.

Theorem 3.1. *There exists a small constant $\delta_0 > 0$ depending only on the dimension d, such that for any $\delta \in [0, \delta_0]$ and any family of immersions $\{\psi^\epsilon\} \subset \mathcal{F}(d, \delta)$, we can find $\{\sigma^\epsilon\} \subset \mathrm{Diff}(\mathbb{R}^d)$ such that the limit of $\psi^\epsilon \circ \sigma^\epsilon$ converges to a Hölder graph system, after passing to subsequences.*

It is proved in [9, 10] that for sufficiently small δ_0, $\mathcal{M}$ is homeomorphic to $\mathbb{R}^d$ and behaves nicely on small scales — for each $x \in \mathcal{M}$ and $R > 0$, $B(x,R) \cap \mathcal{M}$ stays close to the hyperplane through x normal to the averaged Gauss map $\mathbf{n}_{x,R}$. Indeed, $\mathcal{M}$ with small $\|\mathbf{n}\|_{\mathrm{BMO}(\mathcal{M})}$ is equivalent to the definition of a chord-arc surface with small constant, defined in [8] as a generalisation of the chord-arc domain for $d = 1$. Although it remains an open question if such $\mathcal{M}$ always admits bi-Lipschitz parametrisations by $\mathbb{R}^d$ (*cf.* T. Toro [11] for a related problem), it is nevertheless known that $\mathcal{M}$ has a "bi-Hölder" parametrisation; see Theorem 4.1, [9]. This enables us to prove Theorem 3.1.

Proof. Let us first summarise several estimates from [8, 9, 10]. Fix any $t > 0$, *e.g.* $t = 10^{-5}$. By §3, [9] one can find a new chord-arc surface $\mathcal{M}_t$ with the chord-arc constant μ, such that

$$0 \leq \delta \leq \delta_0 \leq C(d)\delta_0 < \mu.$$

We shall choose μ later, which is equivalent to the least upper bound for the BMO-norm of the Gauss map; see p.200 [8]. In view of Eq. (3.7) and Lemma 3.8 in [9], $\mathcal{M}_t \cap B(x, [2^{-1} + 10^{-10}]t)$ is a Lipschitz graph with constant $\leq C_0\mu$ for each $x \in \mathcal{M}$, provided that $\mu = \mu(t, \delta_0)$ is chosen large enough. Here $C_0 = C(d, \delta_0)$. Under the same condition, $\mathcal{M}_t$ can be taken sufficiently close to $\mathcal{M}$ (*e.g.*, with distance $\leq 10^{-10}t$ by Lemma 3.8 in [9]). Then, in view of Theorem 4.1 in [9], there exists a homeomorphism $\tau : \mathcal{M} \to \mathcal{M}_t$ such that

$$\max\left\{\|\tau\|_{C^{0,\gamma}(B(x,100t)\cap\mathcal{M})}, \|\tau^{-1}\|_{C^{0,\gamma}(B(x,100t))\cap\mathcal{M}_t}\right\} \leq C_1 \qquad \text{for all } x \in \mathcal{M}, \tag{3.2}$$

where $C_1 = C(d, \delta_0, t)$ and the Hölder index is given by

$$\gamma \equiv 1 - C_2 d\delta_0 \tag{3.3}$$

for a dimensional constant C_2 (denoted by k in [9]). In fact, putting together Eqs. (1.3)(4.6) and the choice of p on p.178 in [9], Lemma 5.5 in [8] and that $0 \leq \delta \leq \delta_0$, we may explicitly select

$$C_1 = C_3^{C_2\delta_0}\left\{\frac{(100t)^{C_2\delta_0}}{1 - 2\cdot 10^d\delta_0}\right\}. \tag{3.4}$$

Here $C_3 = C_3(d)$ is a dimensional constant. Notice that our estimates (3.4)(3.2) are uniform in δ. We also have to further restrict to $\delta_0 < (C_2 d)^{-1}$ to ensure that $\gamma > 0$ in (3.3).

Now we are ready to give the proof. By considering a compact exhaustion $\{\mathcal{M}_k\} \nearrow \mathcal{M}$, one may take $\mathcal{M}$ to be a bounded domain in $\mathbb{R}^d$. (The argument for non-compact manifolds in the $p > d$ case is more involved, if one needs to check that the limiting object is a manifold; see §7 in [1].) Then we can take a $(50t)$-net $\mathcal{N}$ of $\mathcal{M}$, whose cardinality is

$$\mathcal{H}^0(\mathcal{N}) = C_4 t^{-d}$$

for some geometric constant $C_4 = C(d, \gamma) \equiv C(d, \delta_0)$. In each element of $\mathcal{N}$ the hypersurface $\mathcal{M}$ is $C^{0,\gamma}$-parametrised by $\mathcal{M}_t$, which is a Lipschitz graph on $(2^{-1} + 10^{-10})$-balls. Using the quantitative estimates in the preceding paragraph, we can refine $\mathcal{N}$ to a sub-net $\tilde{\mathcal{N}}$ with cardinality $C_5 t^{-d}$, $C_5 = C(d, \delta_0)$ again, such that in each $B \in \tilde{\mathcal{N}}$, the set $B \cap \mathcal{M}$ is parametrised by a $C^{0,\gamma}$-homeomorphism with the Hölder norm bounded by $C_6 := C_0\mu \cdot C_1$. Let us choose $\mu = 10C(d)\delta_0$; then $C_6 = C(d, \delta_0, t)$ (where $t > 0$ is fixed from the beginning). Therefore, in view of the Arselà–Ascoli theorem, *i.e.*, the compactness of he embedding $C^{0,\gamma} \hookrightarrow C^{0,\gamma'}$ for $\gamma' \in]0, \gamma[$ and the uniform estimates derived above, we may complete the proof. $\square$

4. Two Further Questions

Let the moduli space $\mathcal{F}(A,E,p)$ be as in §1. Is the space

$$\mathcal{F}_{\rm isom}(A,E,p) := \Big\{\psi \in \mathcal{F}(A,E,p) : \psi \text{ is an isometric immersion of a fixed manifold} \mathcal{M}\Big\}$$

compact in its natural topology? For the end-point case $p = 2 = d$ the answer is affirmative, in contrast to the unconstrained case for $\mathcal{F}(A,E,p)$. The authors of [3] proved this via establishing the weak continuity of the Gauss–Codazzi equations (the PDE system for the isometric immersion), with the help of a div-curl type lemma due to Conti–Dolzmann–Müller in [4]. What about higher dimensions $d \geq 3$ (and co-dimensions greater than 1)? That is, for a family of isometric immersions of some fixed d-dimensional manifold with uniformly bounded second fundamental forms in L^d, is the subsequential limit an isometric immersion?

Theorem 3.1 leaves open the possibility that the limiting objects of $W^{2,d}$-bounded immersed hypersurfaces may be very irregular (*e.g.*, the nowhere differentiable Weierstrass function is $C^{0,\gamma}$, or other fractals), even if the (somewhat strong) geometrical condition that the Gauss map is slowly oscillating is enforced. Can we find natural geometrical conditions on the moduli space of d-dimensional hypersurfaces with uniformly bounded second fundamental forms in L^d, which is sufficient to ensure higher regularities for the subsequential limits, *e.g.*, BV or Lipschitz? This is related to the problem of finding good parametrisations of chord-arc surfaces; see the discussions by S. Semmes [9] and T. Toro [11].

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