REGULARITY OF DESINGULARIZED MODELS FOR VORTEX FILAMENTS IN INCOMPRESSIBLE VISCOUS FLOWS: A GEOMETRICAL APPROACH

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Dedicated to Bob Hardt, with Admiration and Friendship

Summary

We establish the regularity of weak solutions for the vorticity equation associated to a family of desingularized models for vortex filament dynamics in 3D incompressible viscous flows. These generalize the classical model 'of an allowance for the thickness of the vortices' due to Louis Rosenhead in 1930. Our approach is based on an interplay between the geometry of vorticity and analytic inequalities in Sobolev spaces.

1. Introduction

1.1 *The PDE for vorticity*

In this article, we study the regularity issues of vortex filament dynamics in an incompressible viscous fluid in $\mathbb{R}^3$. The *vorticity* ω is a vectorfield of physical significance: it measures the 'size of rotations' in the fluid and plays a central rôle in the regularity theory of the fluid flow (*cf.* Wolibner (**1**) and Yudovich (**2**)).

The dynamics of ω is described by the following partial differential equations (PDE):

$$\frac{\partial\omega}{\partial t} + (u\cdot\nabla)\omega - \nu\Delta\omega = \mathbb{S}\cdot\omega \qquad \text{in } [0, T^\star[\times\mathbb{R}^3. \tag{1.1}$$

Here, $\omega : [0, T^\star[\times\mathbb{R}^3 \to \mathbb{R}^3$ is the vorticity, $u : [0, T^\star[\times\mathbb{R}^3 \to \mathbb{R}^3$ is the velocity, the constant $\nu > 0$ is the kinematic viscosity, and $\mathbb{S} = \mathbb{S}(u) : [0, T^\star[\times\mathbb{R}^3 \to \mathbb{R}^3 \otimes \mathbb{R}^3$ is the rate-of-strain tensor of the fluid, defined as

$$\mathbb{S} := \frac{1}{2}\big(\nabla u + \nabla^\top u\big). \tag{1.2}$$

Equation (1.1) is supplemented by the incompressibility condition

$$\operatorname{div} u = 0 \qquad \text{in } [0, T^\star[\times\mathbb{R}^3 \tag{1.3}$$

and the initial condition

$$\omega|\{t = 0\} = \omega_0 \qquad \text{on } \{0\}\times\mathbb{R}^3. \tag{1.4}$$

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1.2 *Vortex filaments*

Our work is concerned with models of line vortices and vortex filaments. Related physical notions and mathematical models are summarized in Sections 1.2–1.4, which are essentially taken from Section 2 in Berselli–Gubinelli (**3**).

A *line vortex* is a singular distribution in which infinite vorticity is concentrated on a curve γ, such that the circulation $\Gamma > 0$ around a closed circuit threaded by γ is finite. Γ is called the *strength* of the line vortex. One may view a line vortex as obtained via the limiting process of pinching a *vortex filament*—that is, a vortex tube surrounded by the fluid—to the curve γ, with the strength being kept constant (*cf.* Helmholtz (**4**)).

We consider the case that the curve supporting the vorticity is a *knot*, that is, a smooth simple closed curve $\gamma : [0, 1] \to \mathbb{R}^3$ with $\gamma(0) = \gamma(1)$. Throughout γ is identified with its image.

Formally, the vorticity vectorfield ω is given by (2.4) in (**3**):

$$\omega(t, x) = \Gamma \int_0^1 \delta\big(x - \gamma(t, \xi)\big)\gamma_\xi(t, \xi)\, d\xi \qquad \text{for each } x \in \mathbb{R}^3,\ t \in [0, T^\star[. \tag{1.5}$$

Here, δ is the Dirac delta function, $\xi \in [0, 1]$ is the arclength parameter, and $\gamma_\xi = \partial\gamma/\partial\xi$. In measure-theoretic notations, we write $\omega(t, x) = \Gamma \int_0^1 \gamma_\xi(t, \xi)\, \mathrm{d}\mu(\xi)$, with the measure μ being the restriction of the one-dimensional Hausdorff measure to the curve: $\mu \equiv \mathcal{H}^1 \llcorner \{x = \gamma(t, \bullet)\}$.

1.3 *Biot–Savart laws*

As is well-known, for the Navier–Stokes and Euler equations, the vorticity ω is related to the velocity u of the fluid via

$$\omega = \operatorname{curl} u = \nabla \wedge u. \tag{1.6}$$

In $\mathbb{R}^3$ one may represent the operator curl^{-1} as a singular integral:

$$u(t, x) = -\frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{x - y}{|x - y|^3} \wedge \omega(t, y)\, dy. \tag{1.7}$$

Such a relation is known as a *Biot–Savart law*.

Consider the case of a line vortex. Assuming that the knot γ is transported by the velocity vectorfield, one may deduce from (1.5) and (1.7) that

$$\frac{\partial\gamma}{\partial t}(t, \xi) = -\frac{\Gamma}{4\pi} \int_0^1 \frac{\gamma(t, \xi) - \gamma(t, \eta)}{|\gamma(t, \xi) - \gamma(t, \eta)|^3} \wedge \gamma_\eta(t, \eta)\, d\eta; \tag{1.8}$$

see (**3**, **5**, **6**). Near the diagonal $\{\xi = \eta\} \subset [0, 1] \times [0, 1]$ the PDE (1.8) is highly singular.

1.4 *Rosenhead approximation*

In 1930, Rosenhead (**7**) proposed and analytically studied a desingularized model for (1.8). The paper (**7**) begins as such:

> This article is an attempt to investigate the effect on the configuration of vortices in the wake behind a cylinder of an allowance for the thickness of the vortices...

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The idea of Rosenhead's approximation is to smear out the singularity of (1.8) on the diagonal by considering the desingularized vortex equation:

$$\frac{\partial \gamma}{\partial t}(t,\xi) = -\frac{\Gamma}{4\pi}\int_0^1 \frac{\gamma(t,\xi)-\gamma(t,\eta)}{\left[\left(\gamma(t,\xi)-\gamma(t,\eta)\right)^2+\mu^2\right]^{3/2}} \wedge \gamma_\eta(t,\eta)\,d\eta \tag{1.9}$$

for some constant $\mu > 0$. See also Moore (**8**) for an application of this model in numerical computations for aircraft trailing vortices.

In effect, one may view Rosenhead's desingularized model (1.9) as obtained via a *modified Biot–Savart law*. As in (**3**, **5**), (1.9) amounts to expressing u in terms of ω by

$$u(t,x) = -\frac{1}{4\pi}\int_{\mathbb{R}^3} \nabla\phi(x-y)\wedge\omega(t,y)\,\mathrm{d}y, \tag{1.10}$$

with the potential $\phi : [0, T^\star[\times\mathbb{R}^3 \to \mathbb{R}$ given by

$$\phi(z) = \frac{\Gamma}{\sqrt{|z|^2+\mu^2}}. \tag{1.11}$$

Note that $\phi(z)$ becomes completely regular as $|z| \to 0$.

One should compare Rosenhead's model with the usual Biot–Savart law for $u = \mathrm{curl}^{-1}\omega$, namely (1.7). The latter equation can be obtained from (1.11) by setting $\mu = 0$.

1.5 *Partial desingularizations*

Our main goal is to investigate 'partially desingularized' models for (1.8). We shall analyze the Biot–Savart law (1.10) with the potential

$$\phi_\delta(z) = \frac{\Gamma}{\sqrt{|z|^2+\mu^2|z|^\delta}} \qquad \text{for a positive parameter } \delta. \tag{1.12}$$

In the case $0 < \delta < 2$, we obtain a desingularized model which is *more singular* than Rosenhead's approximation ($\delta = 0$) in (1.10), (1.11) and (1.9), but *less singular* than $u = \mathrm{curl}^{-1}\omega$ ($\delta = 2$).

Using methods pioneered by Constantin–Fefferman (**9**) in the study of geometrical regularity criteria for the Navier–Stokes equations, we establish the regularity of weak solutions for the vorticity equation (1.1), under the partially desingularized Biot–Savart laws (1.12) with

$$0 \le \delta < 1.$$

The precise statement of our results are given in Section 2.

1.6 *Related works*

For the background on the mathematical analysis of fluid dynamical PDEs, we refer to Constantin–Foias (**9**), Temam (**10**), Ladyzhenskaya (**11**), Lemarié-Rieusset (**12**), Seregin (**6**), Galdi (**13**), Robinson–Rodrigo–Sadowski (**14**), and many others.

The dynamics of vortices is an important topic in aero- and hydro-dynamics; see Saffman (**6**) and Chorin (**15**) for a comprehensive treatment. The existence, uniqueness and stability properties of various PDE models for vortex dynamics have been studied; *cf.* Banica–Vega (**16**), Aiki–Iguchi (**17**), Jerrard–Seis (**18**), Lions–Majda (**19**), etc. The study of vortex dynamics in incompressible *viscous* fluid flows attracts much attention in recent works; *cf.* Enciso–Lucà–Peralta-Salas (**20**).

Regarding the Rosenhead model, a rigorous analytic study was first carried out by Berselli–Bessaih (**5**). It is extended to more general stochastic contexts by Bessaih–Gubinelli–Russo (**21**); see also Flandoli (**22**).

The global existence of the smooth solution for (1.1) and (1.12) was proved by Berselli–Gubinelli (**3**) under a few mild assumptions on the *Fourier side* of the potential ϕ in the Biot–Savart law; see 'Hypothesis A' on p.698 therein. The results in (**3**) cover a wide range of desingularized models for vortex filament dynamics with modified Biot–Savart laws, including the Rosenhead approximation. In this article, we are concerned with desingularized models of a different nature: the desingularization effects take place on the *physical side* of ϕ. The potentials considered in Section 1.5 (see (1.12)) appear to be still more singular at the singularity $z = 0$.

The key new feature of this article is to apply ideas and techniques from the works on *geometric regularity criteria* for the Navier–Stokes equations to analyze the desingularized models for vortex filament dynamics. First proposed by Constantin–Fefferman (**9**), the geometric regularity criteria can be summarized as follows (here the vorticity and velocity are related by the usual Biot–Savart law (1.7)): 'If for all $t \in [0, T^\star[$ the angle between the vorticity vectorfield of a weak solution for the Navier–Stokes equations at nearby points satisfies certain uniform Hölder conditions in space, then it is automatically strong up to the time $T^\star$'. Such criteria have been further developed in many works; see, for example, (**13**, **23**–**38**). This list is by no means exhaustive.

Finally, intriguing linkages between topological/differential geometric works on *knot energies* and analytic studies on line vortex dynamics deserve further explorations. Such linkages are suggested by (1.8). We refer to Freedman–He–Wang (**39**) and O'Hara (**40**) for knot energies.

2. Main result

2.1 *Theorem*

The main result of our article is as follows. For the convenience of readers, we reproduce (1.1), (1.10) and (1.12) in the statement below.

> Let $\omega \in L^\infty(0, T^\star; L^1 \cap W^{-1,2}(\mathbb{R}^3; \mathbb{R}^3)) \cap L^2(0, T^\star; L^2(\mathbb{R}^3; \mathbb{R}^3))$ be a weak solution for the vorticity equation:
>
> $$\frac{\partial \omega}{\partial t} + (u \cdot \nabla)\omega - \nu\Delta\omega = \mathbb{S} \cdot \omega \qquad \text{in } [0, T^\star[\times\mathbb{R}^3.$$
>
> Assume that u is related to ω via the modified Biot–Savart law:
>
> $$u(t, x) = -\frac{1}{4\pi}\int_{\mathbb{R}^3} \nabla\phi_\delta(x - y) \wedge \omega(t, y)\, dy$$
>
> where, for positive constants Γ and μ,
>
> $$\phi_\delta(z) = \frac{\Gamma}{\sqrt{|z|^2 + \mu^2|z|^\delta}}.$$

Furthermore, assume that $\{\omega(t, \bullet)\}_{t\in[0,T^\star[}$ is supported on a compact set in $\mathbb{R}^3$. Then, the vorticity ω has higher regularity as follows: $\omega \in L^\infty(0, T^\star; L^2(\mathbb{R}^3; \mathbb{R}^3)) \cap L^2(0, T^\star; W^{1,2}(\mathbb{R}^3; \mathbb{R}^3))$ whenever $\delta \in [0, 1[$.

2.2 *Remark*

The assumption $\omega \in L^\infty(0, T^\star; L^1(\mathbb{R}^3; \mathbb{R}^3))$ is due to the uniform boundedness of total circulation. We put

$$\Sigma_0 := \sup_{0\le t<T^\star} \int_{\mathbb{R}^3} |\omega(t, x)|\, \mathrm{d}x < \infty. \tag{2.1}$$

Compactness of the support for $\{\omega(t, \bullet)\}_{t\in[0,T^\star[}$ means that the vortex filament does not become infinitely large up to time $T^\star$. These agree with the discussions in Section 1.2 on the physical model.

In this article, weak solutions are understood in the distributional sense as usual; for example, for the vorticity equation (1.1), one needs to integrate against arbitrary test functions of the form $\psi(t)\chi(x)$ for $\phi \in C_0^\infty(]-1, T^\star[)$ and $\chi \in C_c^\infty(\mathbb{R}^3)$. The local-in-time existence of weak solutions can be obtained by adapting the Galerkin approximation scheme for the Navier–Stokes equations; see, for example, Temam (**10**) Chapter III Section 3.

2.3 *Strategy of the proof*

We represent the rate-of-strain tensor $\mathbb{S} = \mathbb{S}(u)$ as a singular integral of ω via the modified Biot–Savart law. Then, we show that the vorticity stretching term

$$\mathfrak{S}(t) := \int_{\mathbb{R}^3} \mathbb{S}(t, x) : \omega(t, x) \otimes \omega(t, x)\, dx \tag{2.2}$$

can be controlled by the *enstrophy*

$$\mathcal{E}(t) := \frac{1}{2} \int_{\mathbb{R}^3} |\omega(t, x)|^2\, dx. \tag{2.3}$$

(Throughout, for 3×3 matrices $\mathfrak{n}$ and $\mathfrak{p}$ we write $\mathfrak{n} : \mathfrak{p} \equiv \mathrm{tr}(\mathfrak{n}^\top \mathfrak{p})$.) This is done by exploring the rôle of the angle $\angle(\omega(t, x), \omega(t, y))$ played in the singular integral as in (**9**).

3. Inequalities

In this section, we summarize a few well-known analytic inequalities.

3.1 *Interpolation for L^p*

Let p_0, p_1 be such that $0 < p_0 < p_1 \le \infty$. For any $0 \le \theta \le 1$ define p_θ by $\frac{1}{p_\theta} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$. Then, for any $n = 1, 2, 3, \ldots$ and any $f \in L^{p_0}(\mathbb{R}^n) \cap L^{p_1}(\mathbb{R}^n)$, there holds

$$\|f\|_{L^{p_\theta}(\mathbb{R}^n)} \le \|f\|^{1-\theta}_{L^{p_0}(\mathbb{R}^n)} \|f\|^{\theta}_{L^{p_1}(\mathbb{R}^n)}.$$

3.2 *Gagliardo–Nirenberg–Sobolev inequality*

Let $f : \mathbb{R}^n \to \mathbb{R}$. Fix $1 \le q, r < \infty$ and $m = 1, 2, 3, \ldots$. Suppose that $\alpha \in \mathbb{R}$ and $j \in \mathbb{N}$ satisfy

$$\frac{j}{m} \le \alpha \le 1, \qquad \frac{1}{p} = \frac{j}{n} + \Big(\frac{1}{r} - \frac{m}{n}\Big)\alpha + \frac{1-\alpha}{q}.$$

Then there exists a constant C depending only on m, n, j, q, r and α such that

$$\|D^j f\|_{L^p(\mathbb{R}^n)} \le C\|D^m f\|^{\alpha}_{L^r(\mathbb{R}^n)}\|f\|^{1-\alpha}_{L^q(\mathbb{R}^n)}.$$

3.3 *Hardy–Littlewood–Sobolev inequality*

Let $f \in L^p(\mathbb{R}^n)$ and $g \in L^s(\mathbb{R}^n)$ with $1 < p, s < \infty$. Assume that $0 < \lambda < n$ satisfies $1/p + 1/s + \lambda/n = 2$. Then there is a constant C depending only on p, n and λ such that

$$\left| \iint_{\mathbb{R}^n \times \mathbb{R}^n} f(x)|x-y|^{-\lambda} g(y)\, dx\, dy \right| \le C\|f\|_{L^p(\mathbb{R}^n)}\|g\|_{L^s(\mathbb{R}^n)}.$$

4. Proof of 2.1 theorem

4.1 *Preliminary energy estimate*

Multiplying ω to both sides of the vorticity equation (1.1) and integrating over space, we obtain

$$\frac{d\mathcal{E}}{dt}(t) + \nu \int_{\mathbb{R}^3} |\nabla\omega(t,x)|^2\, dx = \mathfrak{S}(t).$$

The right-hand side is the vorticity stretching term given by (2.2). One should note that this evolution equation for $\mathcal{E}$ is understood in the sense of distributions over $[0, T^\star[$.

4.2 *Singular integral representation of* $\mathbb{S}$

Now we show

Lemma. *In the setting of Theorem 2.1, the rate-of-strain tensor* $\mathbb{S}$ *can be represented as*

$$\mathbb{S}(x) = \text{p.v.} \int_{\mathbb{R}^3} \left\{ \frac{\Gamma\delta(2-\delta)}{4}\mu^2 A^{-\frac{3}{2}}(|x-y|)\cdot|x-y|^{\delta-4} + \frac{3\Gamma}{8}B^2(|x-y|)A^{-\frac{5}{2}}(|x-y|) \right\} \cdot \cdot \left\{ (x-y)\wedge\omega(y)\otimes(x-y) + (x-y)\otimes(x-y)\wedge\omega(y) \right\} dy. \tag{4.1}$$

Here $A, B : [0,\infty[\to [0,\infty[$ *are given by*

$$A(r) := r^2 + \mu^2 r^\delta, \tag{4.2}$$

$$B(r) := 2 + \delta\mu^2 r^{\delta-2}. \tag{4.3}$$

Most estimates in this paper hold pointwise in t; so, unless otherwise declared, we always suppress the t-variable. Throughout $p.v.$ denotes the principal value of singular integrals; for vectors $a = (a^1, a^2, a^3)$ and $b = (b^1, b^2, b^3) \in \mathbb{R}^3$, $a \otimes b$ is the 3×3 matrix $\{a^i b^j\}_{1 \le i,j \le 3}$.

4.3 *Proof of 4.2 Lemma*

First let us show that

$$\nabla u(x) = \text{p.v.} \int_{\mathbb{R}^3} \nabla\nabla\phi_\delta(x-y) \wedge \omega(y)\, dy, \tag{4.4}$$

where

$$(\nabla\nabla\phi_\delta \wedge \omega)_i^j \equiv \nabla_i(\nabla\phi_\delta \wedge \omega)^j$$

for $i, j \in \{1, 2, 3\}$. This is an equality for 3×3 matrices.

Indeed, direct computation yields that

$$\begin{aligned}\nabla\phi_\delta(z) &= -\frac{\Gamma}{2}\nabla\Big(|z|^2 + \mu^2|z|^\delta\Big)\Big(|z|^2 + \mu^2|z|^\delta\Big)^{-\frac{3}{2}} \\ &= -\frac{\Gamma}{2}\Big(2z + \delta\mu^2|z|^{\delta-2}z\Big)\Big(|z|^2 + \mu^2|z|^\delta\Big)^{-\frac{3}{2}}.\end{aligned} \tag{4.5}$$

It is locally integrable on $\mathbb{R}^3$. This enables us to compute the weak Hessian $\nabla\nabla\phi_\delta$ via the dominated convergence theorem as follows.

Take an arbitrary test function $\chi \in C_c^\infty(\mathbb{R}^3)$. The above paragraph ensures that

$$-\langle \chi, \nabla\nabla\phi_\delta \rangle = \lim_{\epsilon \searrow 0} \int_{\{|x| \geq \epsilon\}} \nabla\phi_\delta(x) \otimes \nabla\chi(x)\, dx,$$

where the left-hand side is the paring of a distribution with a test function. In light of integration by parts and the divergence theorem, it further equals

$$\lim_{\epsilon \searrow 0} \left\{ -\int_{\{|x| \geq \epsilon\}} \nabla\nabla\phi_\delta(x)\chi(x)\, dx + \int_{\{|x| = \epsilon\}} \chi(x)\nabla\phi_\delta(x) \otimes \frac{x}{|x|}\, d\mathcal{H}^2(x) \right\}.$$

For the second term, a change of variable leads to

$$\int_{\{|x|=\epsilon\}} \chi(x)\nabla\phi_\delta(x) \otimes \frac{x}{|x|}\, d\mathcal{H}^2(x) = \epsilon^2 \int_{\{|x|=1\}} \chi(\epsilon x)\nabla\phi_\delta(\epsilon x) \otimes x\, d\mathcal{H}^2(x).$$

By the definition of ϕ_δ in (1.12), the right-hand side is controlled by $C(\mu, \Gamma)\epsilon^{2-\frac{3\delta}{2}}\|\chi\|_{L^\infty(\mathbb{R}^3)}$. For any $\delta \in [0, 1[$ this tends to zero as $\epsilon \searrow 0$; so (4.4) follows.

The previous arguments justify the computation of $\nabla\nabla\phi_\delta$ by directly taking ∇ to the final line in (4.5). In local coordinates, we get

$$\begin{aligned}\nabla_i\nabla_j\phi_\delta(z) &= -\frac{\Gamma}{2}\nabla_i\left\{\frac{2z^j + \delta\mu^2|z|^{\delta-2}z^j}{A^{\frac{3}{2}}(|z|)}\right\} \\ &= -\frac{\Gamma}{2}A^{-3}(|z|)\bigg\{A^{\frac{3}{2}}(|z|)\Big[2\tilde\delta_{ij} + \delta\mu^2\tilde\delta_{ij}|z|^{\delta-2} + \delta(\delta-2)\mu^2|z|^{\delta-4}z^iz^j\Big] \\ &\qquad - \frac{3}{2}\Big(2z^j + \delta\mu^2z^j|z|^{\delta-2}\Big(A^{\frac{1}{2}}(|z|)\nabla_iA(|z|)\Big)\bigg\}.\end{aligned}$$

Here, $\tilde\delta_{ij}$ is the Kronecker delta symbol (lest it gets confused with the parameter $\delta \in [0, 1[$).

Next, note the simple identity

$$\nabla_i A(|z|) = z^i B(|z|),$$

from which we infer that

$$\nabla\nabla\phi_\delta(z) = -\frac{\Gamma}{2}A^{-\frac{3}{2}}(|z|)A^{-\frac{3}{2}}(|z|)\Big[\delta\mu^2\tilde{\delta} + \delta(\delta-2)\mu^2|z|^{\delta-4}z\otimes z\Big] + \frac{3\Gamma}{4}A^{-\frac{5}{2}}(|z|)B^2(|z|)z\otimes z.$$

Thus, for some *anti-symmetric* matrix $\mathfrak{m} \in \mathfrak{so}(3,\mathbb{R})$, there holds

$$\begin{aligned}&\nabla\nabla\phi_\delta(x-y)\wedge\omega(y)\\ &\quad= \mathfrak{m} + \Big\{(x-y)\wedge\omega(y)\otimes(x-y)\Big\}\cdot\\ &\qquad\cdot\Big\{-\frac{\Gamma}{2}\delta(\delta-2)\mu^2A^{-\frac{3}{2}}(|x-y|)|x-y|^{\delta-4} + \frac{3\Gamma}{4}A^{-\frac{5}{2}}(|x-y|)B^2(|x-y|)\Big\}. \qquad (4.6)\end{aligned}$$

We substitute (4.6) into (4.4) to get the singular integral representation for ∇u. The lemma in Section 4.2 follows immediately by symmetrizing the resulting expression.

4.4 *Vorticity stretching term $\mathfrak{S}$*

The singular integral representation for $\mathbb{S}$ in Section 4.2 implies

Lemma. *The vorticity stretching term $\mathfrak{S} = \int_{\mathbb{R}^3}\mathbb{S} : \omega\otimes\omega\, dx$ can be bounded as follows:*

$$|\mathfrak{S}| \leq \int_{\mathbb{R}^3}|\omega(x)|^2\Big\{\int_{\mathbb{R}^3}\Big[K^{(1)}(|x-y|)+K^{(2)}(|x-y|)\Big]|\omega(y)|\,dy\Big\}\,dx, \qquad (4.7)$$

where

$$K^{(1)}(r) := \frac{\Gamma\delta(2-\delta)}{4}\mu^2r^{\delta-2}A^{-\frac{3}{2}}(r), \qquad (4.8)$$

$$K^{(2)}(r) := \frac{3\Gamma}{8}r^2B^2(r)A^{-\frac{5}{2}}(r). \qquad (4.9)$$

4.5 *Proof of 4.4 Lemma*

In Section 4.2, we proved that

$$\begin{aligned}\mathbb{S}(x) = \text{p.v.}\int_{\mathbb{R}^3}&\Big[K^{(1)}(|x-y|)+K^{(2)}(|x-y|)\Big]\cdot\\ &\cdot\Big\{(x-y)\wedge\omega(y)\otimes(x-y)+(x-y)\otimes(x-y)\wedge\omega(y)\Big\}\,dy.\end{aligned}$$

Following the crucial observations due to Constantin–Fefferman (**9**), by writing

$$\widehat{z} := \frac{z}{|z|} \qquad \text{for any } z \in \mathbb{R}^3,$$

one may express the vorticity stretching term as

$$\begin{aligned}\mathfrak{S} = \int_{\mathbb{R}^3} |\omega(x)|^2 \Bigg\{ \int_{\mathbb{R}^3} |\omega(y)| \Big[K^{(1)}(|x-y|) + K^{(2)}(|x-y|) \Big] \cdot \\ \cdot \begin{bmatrix} \widehat{x-y} \wedge \widehat{\omega}(y) \otimes \widehat{x-y} \\ + \\ \widehat{x-y} \otimes \widehat{x-y} \wedge \widehat{\omega}(y) \end{bmatrix} : \Big[\widehat{\omega}(x) \otimes \widehat{\omega}(x) \Big] \Bigg\} \, dx.\end{aligned}$$

But

$$\begin{bmatrix} \widehat{x-y} \wedge \widehat{\omega}(y) \otimes \widehat{x-y} \\ + \\ \widehat{x-y} \otimes \widehat{x-y} \wedge \widehat{\omega}(y) \end{bmatrix} : \Big[\widehat{\omega}(x) \otimes \widehat{\omega}(x) \Big] = \mathcal{D}\Big(\widehat{x-y}, \widehat{\omega}(x), \widehat{\omega}(y) \Big),$$

where for arbitrary unit column vectors $e_1, e_2, e_3 \in \mathbb{R}^3$ we write

$$\mathcal{D}(e_1, e_2, e_3) := e_1 \cdot e_3 \det(e_1|e_2|e_3).$$

Thus elementary calculus gives us the pointwise estimate:

$$\Big| \mathcal{D}\Big(\widehat{x-y}, \widehat{\omega}(x), \widehat{\omega}(y) \Big) \Big| \le \Big| \sin \angle \Big(\omega(t,x), \omega(t,y) \Big) \Big|. \tag{4.10}$$

We can now conclude 4.4 by naïvely bounding this term by 1.

4.6 *Conclusion of the proof*

We resume from estimate (4.7) for the vorticity stretching term $\mathfrak{S}$. Thanks to (4.2), (4.3), (4.8) and (4.9), pointwise we have

$$K^{(1)}(|z|) \le \frac{\Gamma\delta(2-\delta)}{4\mu} |z|^{-2-\frac{\delta}{2}}, \tag{4.11}$$

$$K^{(2)}(|z|) \le \frac{3\Gamma\delta^2}{2\mu} |z|^{-2-\frac{\delta}{2}} \qquad \text{if } |z| \le \left(\frac{\delta\mu^2}{2} \right)^{\frac{1}{2-\delta}}, \tag{4.12}$$

$$K^{(2)}(|z|) \le \frac{6\Gamma}{\mu^5} |z|^{2-\frac{5\delta}{2}} \qquad \text{if } |z| \ge \left(\frac{\delta\mu^2}{2} \right)^{\frac{1}{2-\delta}}. \tag{4.13}$$

Indeed, for $K^{(1)}$ we may bound

$$\begin{aligned} K^{(1)}(|z|) &\le \frac{\Gamma\delta(2-\delta)}{4} \frac{\mu^2 |z|^{\delta-2}}{\left(|z|^2 + \mu^2 |z|^{\delta}\right)^{\frac{3}{2}}} \\ &\le \frac{\Gamma\delta(2-\delta)}{4} \frac{\mu^2 |z|^{\delta-2}}{\left(\mu^2 |z|^{\delta}\right)^{\frac{3}{2}}} = \frac{\Gamma\delta(2-\delta)}{4\mu} |z|^{-2-\frac{\delta}{2}}. \end{aligned}$$

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For $K^{(2)}$, when $|z| \geq (\delta\mu^2/2)^{\frac{1}{2-\delta}}$ one has $\delta\mu^2|z|^{\delta-2} \leq 2$. It implies that

$$K^{(2)}(|z|) \leq \frac{3\Gamma}{8}\frac{|z|^2(2+\delta\mu^2|z|^{\delta-2})^2}{\left(|z|^2+\mu^2|z|^\delta\right)^{\frac{5}{2}}}$$
$$\leq 6\Gamma\frac{|z|^2}{\left(|z|^2+\mu^2|z|^\delta\right)^{\frac{5}{2}}} \leq 6\Gamma\frac{|z|^2}{(\mu^2|z|^\delta)^{\frac{5}{2}}} = \frac{6\Gamma}{\mu^5}|z|^{2-\frac{5\delta}{2}}.$$

On the other hand, when $|z| \leq (\delta\mu^2/2)^{\frac{1}{2-\delta}}$ there holds $\delta\mu^2|z|^{\delta-2} \geq 2$. So one infers that

$$K^{(2)}(|z|) \leq \frac{3\Gamma}{8}\frac{|z|^2(2+\delta\mu^2|z|^{\delta-2})^2}{\left(|z|^2+\mu^2|z|^\delta\right)^{\frac{5}{2}}}$$
$$\leq \frac{3\Gamma}{8}\frac{|z|^2\cdot 4\delta^2\mu^4|z|^{2\delta-4}}{\left(|z|^2+\mu^2|z|^\delta\right)^{\frac{5}{2}}} \leq \frac{3\Gamma}{8}\frac{|z|^2\cdot 4\delta^2\mu^4|z|^{2\delta-4}}{(\mu^2|z|^\delta)^{\frac{5}{2}}} = \frac{3\Gamma\delta^2}{2\mu}|z|^{-2-\frac{\delta}{2}}.$$

Let us also introduce the constants

$$\eta := \max\left\{\frac{6\Gamma}{\mu^5}, \frac{\Gamma\delta(2-\delta)}{4\mu}, \frac{3\Gamma\delta^2}{2\mu}\right\}, \tag{4.14}$$

$$\Lambda := \operatorname{diam}^{2-\frac{5\delta}{2}}\left(\operatorname{spt}\{\omega(t,\bullet)\}_{t\in[0,T^\star[}\right). \tag{4.15}$$

Applying bounds (4.11), (4.12) and (4.13) on the kernels to (4.7) in Section 4.4, we deduce that

$$|\mathfrak{S}| \leq \eta\Lambda\|\omega\|_{L^2}^2\|\omega\|_{L^1} + \eta\iint_{\mathbb{R}^3\times\mathbb{R}^3}|\omega(x)|^2|\omega(y)||x-y|^{-2-\frac{\delta}{2}}\,dx\,dy. \tag{4.16}$$

Now we invoke the Hardy–Littlewood–Sobolev inequality in 3.2 and take $p = 1+\epsilon$, $n = 3$, and $s = 2+\frac{\delta}{2}$ therein, for some $\epsilon > 0$ to be determined. Then

$$\left|\iint_{\mathbb{R}^3\times\mathbb{R}^3}|\omega(x)|^2|\omega(y)||x-y|^{-2-\frac{\delta}{2}}\,dx\,dy\right| \leq C_1\|\omega\|^2_{L^{2(1+\epsilon)}(\mathbb{R}^3)}\|\omega\|_{L^s(\mathbb{R}^3)}, \tag{4.17}$$

where C_1 depends only on δ and ϵ, with the index $s = \frac{6(1+\epsilon)}{2(1+4\epsilon)-\delta(1+\epsilon)}$.

Recall that $\delta \in [0,1[$; hence, by additionally requiring that $\epsilon \in]0,\frac{1}{2}[$, we have $s \in]\frac{6}{4-\delta},\frac{6}{2-\delta}[$. So we can fix an ϵ (depending only on δ) in $]0,\frac{1}{2}[$ once and for all to warrant that $1 < s < 2$. Consequently, one may view s as being fixed from now on. As a result, the constant C_1 in the inequality (4.17) depends only on δ. Then the interpolation in 3.1 gives us

$$\|\omega\|_{L^s(\mathbb{R}^3)} \leq \|\omega\|_{L^1(\mathbb{R}^3)}^{\frac{2}{s}-1}\|\omega\|_{L^2(\mathbb{R}^3)}^{2-\frac{2}{s}}. \tag{4.18}$$

On the other hand, the Gagliardo–Nirenberg–Sobolev inequality in 3.2 implies that

$$\|\omega\|^2_{L^{2(1+\epsilon)}(\mathbb{R}^3)} \leq C_2 \|\nabla\omega\|^{\frac{3\epsilon}{1+\epsilon}}_{L^2(\mathbb{R}^3)} \|\omega\|^{\frac{2-\epsilon}{1+\epsilon}}_{L^2(\mathbb{R}^3)}. \tag{4.19}$$

Here C_2 depends only on ϵ, hence only on δ as in the previous paragraph.

Putting together (4.17), (4.18) and (4.19), one obtains

$$\left| \iint_{\mathbb{R}^3\times\mathbb{R}^3} |\omega(x)|^2 |\omega(y)| |x-y|^{-2-\frac{\delta}{2}}\, dx\, dy \right| \leq C_3 \|\nabla\omega\|^{\frac{3\epsilon}{1+\epsilon}}_{L^2(\mathbb{R}^3)} \|\omega\|^{2-\frac{2}{s}+\frac{2-\epsilon}{1+\epsilon}}_{L^2(\mathbb{R}^3)},$$

where $C_3 = C_1 C_2 (\Sigma_0)^{\frac{2}{s}-1}$ depends only on δ and the finite total circulation Σ_0 (see (2.1)).

In addition, the simple inequality $ab \leq \frac{a^p}{p} + \frac{b^q}{q}$ for $\frac{1}{p}+\frac{1}{q} = 1$, $1 < p < \infty$ and $a, b \geq 0$ implies that

$$\left| \iint_{\mathbb{R}^3\times\mathbb{R}^3} |\omega(x)|^2 |\omega(y)| |x-y|^{-2-\frac{\delta}{2}}\, dx\, dy \right| \leq \frac{\nu}{2} \|\nabla\omega\|^2_{L^2(\mathbb{R}^3)} + C_4 \|\omega\|^{2\kappa}_{L^2(\mathbb{R}^3)}, \tag{4.20}$$

where the constant

$$C_4 = \frac{2-\epsilon}{2(1+\epsilon)} \frac{(C_3)^{\frac{2(1+\epsilon)}{3\epsilon}}}{\left(\frac{\nu}{2}\right)^{\frac{3\epsilon}{2-\epsilon}}}, \tag{4.21}$$

and the exponent

$$\kappa = \frac{(1+\epsilon)}{2-\epsilon}\Big(2 - \frac{2}{s} + \frac{2-\epsilon}{1+\epsilon}\Big).$$

Since $s \in]1,2[$ and $\epsilon \in]0, \frac{1}{2}[$ (thus $\frac{1+\epsilon}{2-\epsilon} \in]\frac{1}{2}, 1[$), it is crucial to see that

$$\frac{1}{4} < \kappa < 2, \tag{4.22}$$

where κ depends only on δ.

To conclude, we substitute (4.20) and (4.16) into the energy estimate in Section 4.1 to get

$$\frac{d\mathcal{E}}{dt} + \frac{\nu}{2}\|\nabla\omega\|^2_{L^2(\mathbb{R}^3)} \leq \eta\Lambda\Sigma_0\mathcal{E}(t) + \eta C_4 \mathcal{E}^\kappa \tag{4.23}$$

for the enstrophy $\mathcal{E}(t) = \int_{\mathbb{R}^3} |\omega(t,x)|^2\, dx$. But the hypotheses of the theorem require that $\mathcal{E} \in L^2([0,T^\star[)$. Hence, in light of (4.22), the differential inequality (4.23) implies that $\mathcal{E} \in L^\infty([0,T^\star[)$ and that $\nabla\omega \in L^2([0,T^\star[\times\mathbb{R}^3; \mathbb{R}^3\otimes\mathbb{R}^3)$. This concludes the proof.

5. Concluding remarks

A closer examination on the dependence of constants (see, for example, (4.14), (4.15), (4.21) and (4.23)) shows that we have obtained an upper bound for

$$\text{ess sup}_{t\in[0,T^\star[} \|\omega(t,\bullet)\|^2_{L^2(\mathbb{R}^3)}.$$

This bound is proportional to the strength Γ of vortex filaments, a positive power of the diameter of the support of ω, the total circulation Σ_0, a positive power of $T^\star$ and a positive power of the initial enstrophy $\int_{\mathbb{R}^3} |\omega_0|^2 \, dx$. Also, the bound is *inverse proportional* to μ^5 and ν^{ς}, where $\varsigma = \frac{3\epsilon}{2-\epsilon} \in]0, 1[$ and $0 < \mu \ll 1$ (μ is the regularization parameter).

Nevertheless, we cannot directly pass to the inviscid limit by sending $\epsilon \searrow 0$ (hence $\varsigma \searrow 0$): it corresponds to the endpoint case of the Hardy–Littlewood–Sobolev inequality ($p = 1$ in 3.3).

Meanwhile, we have obtained an upper bound for

$$\|\nabla\omega(t, \bullet)\|^2_{L^2(\mathbb{R}^3)},$$

which have the same dependence on all the parameters as for $\operatorname{ess\,sup}_{t\in[0,T^\star[} \|\omega(t, \bullet)\|^2_{L^2(\mathbb{R}^3)}$, except that it is inverse proportional to $\nu^{1+\varsigma}$.

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